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Conceptual Structure and the Procedural Affordances of Rational Numbers: Relational Reasoning With Fractions and Decimals

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The standard number system includes several distinct types of notations, which differ conceptually and afford different procedures. Among notations for rational numbers, the bipartite format of fractions (*a/b*) enables them to represent 2-dimensional relations between sets of discrete (i.e., countable) elements (e.g., red marbles/all marbles). In contrast, the format of decimals is inherently 1-dimensional, expressing a continuous-valued magnitude (i.e., proportion) but not a 2-dimensional relation between sets of countable elements. Experiment 1 showed that college students indeed view these 2-number notations as conceptually distinct. In a task that did not involve mathematical calculations, participants showed a strong preference to represent partitioned displays of discrete objects with fractions and partitioned displays of continuous masses with decimals. Experiment 2 provided evidence that people are better able to identify and evaluate ratio relationships using fractions than decimals, especially for discrete (or discretized) quantities. Experiments 3 and 4 found a similar pattern of performance for a more complex analogical reasoning task. When solving relational reasoning problems based on discrete or discretized quantities, fractions yielded greater accuracy than decimals; in contrast, when quantities were continuous, accuracy was lower for both symbolic notations. Whereas previous research has established that decimals are more effective than fractions in supporting magnitude comparisons, the present study reveals that fractions are relatively advantageous in supporting relational reasoning with discrete (or discretized) concepts. These findings provide an explanation for the effectiveness of natural frequency formats in supporting some types of reasoning, and have implications for teaching of rational numbers.

Keywords: mathematical reasoning, rational numbers, analogy, concepts, relations

Mathematical Thinking as Modeling

Mathematical understanding fundamentally involves grasping that mathematics is a system of quantitative relations among concepts. It has been argued that a core problem with math education, particularly in the United States, is that greater focus is placed on execution of mathematical procedures than on understanding of quantitative relations (Gonzales et al., 2008; Richland, Stigler, & Holyoak, 2012; Stigler & Hiebert, 1999; Rittle-Johnson & Star, 2007). Nonetheless, educators do make an effort to convey the conceptual structure of mathematics, typically by presenting students with examples of analogous real-world situations. For example, the addition operation is often illustrated with examples of combining subsets of a superset category (e.g., 3 red + 5 blue marbles = 8 marbles). Adults prefer the operation of addition to be applied to sets of objects drawn from a common immediate superordinate (e.g., marbles) or from closely related cohyponyms (e.g., roses and daisies); in contrast, they expect division to be applied to sets that have a functional relationship (e.g., 9 cookies divided among 3 children). Such expectancies suggest that people use a process of semantic alignment (akin to analogical reasoning) to systematically connect arithmetic operations with conceptual categories (Bassok, Chase, & Martin, 1998; Bassok, Pedigo, & Oskarsson, 2008).

Other work also suggests that mathematical thinking involves various forms of mapping between numbers and concepts. By far, the most attention has been directed at the unidimensional concept of magnitude, which, in humans and other primates, enables a mapping from numerical quantities onto a mental (and neural) number line (e.g., Cantlon, Brannon, Carter, & Pelphrey, 2006;

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Dehaene & Changeux, 1993; Moyer & Landauer, 1967; Pinel, Piazza, Bihan, & Dehaene, 2004; for recent theoretical work on the acquisition of magnitudes, see Chen, Lu, & Holyoak, 2014). Another type of mapping links variations in symbolic notation at a perceptual level (e.g., spacing of operator symbols) with relations among procedures (e.g., order of operations; Braithwaite & Goldstone, 2013; Landy, Brookes, & Smout, 2014; Landy & Goldstone, 2007; Zahner & Corter, 2010). More generally, procedures appear to derive meaning from their place in a conceptual structure, rather than standing in isolation as rote routines (e.g., English & Halford, 1995; Hinsley, Hayes, & Simon, 1977; Kintsch & Greeno, 1985; Mochon & Sloman, 2004).

Previous psychological work on mathematics as a form of modeling has focused either on the unidimensional concept of number magnitude, or on relations involving more complex mathematical expressions, such as equations. In the present study we focus on the relational structure of numbers themselves. Whereas all numbers express a unidimensional magnitude, some notations for numbers also inherently represent multidimensional relationships. Such multidimensionality is apparent in numbers introduced in advanced mathematics (e.g., complex numbers); however, we argue it is also a property of relatively simple notations for rational numbers taught in elementary school—most notably, common fractions[1] (e.g., 2/3, 13/4).

Acquisition of Fractions and Decimals

Our focus in this study is on how well-educated reasoners understand fractions and decimals; however, developmental work sheds important light on the difficulties associated with each notation type. After the familiar natural numbers, fractions and then decimals are typically introduced in school (in that order, at least in the curricula generally adopted in the United States) as new types of numbers that can express magnitudes less than one. Both symbolic notations often prove problematic for students. Children, and even some adults, exhibit misconceptions about the complex conceptual structure of fractions (Siegler, Thompson, & Schneider, 2011, 2013; Ni & Zhou, 2005; Stigler, Givvin, & Thompson, 2010). Students often find it difficult to reconcile this perceptually and conceptually unfamiliar type of number with their well-established concept of natural numbers (Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2004), particularly in understanding how the whole numbers within the fraction contribute to its overall magnitude. Instead of processing fractions as one integrated quantity, both children and adults often process the natural number parts of the fraction separately (Kallai & Tzelgov, 2009; Bonato, Fabbri, Umiltà, & Zorzi, 2007). Despite (or perhaps because of) their associated difficulties, an understanding of fractions appears to be very important for subsequent learning. For example, students' ability to place fractions on a number line is uniquely related to their general arithmetic ability and later performance (Siegler, Thompson, & Schneider, 2011, 2012). Students also encounter problems in learning to understand decimals (Rittle-Johnson, Siegler, & Alibali, 2001), but generally master the magnitudes of decimals before fractions (Iuculano & Butterworth, 2011). The relative ease of learning magnitudes of decimals presumably reflects the fact that their implicit denominator is a constant (base 10), rather than a variable, so that a decimal inherently expresses the unidimensional concept of magnitude (Halford, Wilson, & Phillips, 1998, 2010).

Bonato et al. (2007) argue that adults interpret fractions in terms of their component integer parts, suggesting that the natural alignment of fractions may resemble that of integers (i.e., a preference for discrete quantities). But fractions may also be interpreted as holistic numbers, providing a possible basis for them to align with continuous quantities. Schneider and Siegler (2010) showed that when comparing fraction magnitudes, adults show a distance effect (response times decrease as the difference between magnitudes increases), analogous to that found with natural numbers. However, DeWolf, Grounds, Bassok, and Holyoak (2014) found that while the pattern of response times for comparing decimals is almost identical to that for comparing multidigit natural numbers, the pattern for fraction comparisons is dramatically slower, and shows a greatly exaggerated distance effect, indicative of less precise magnitude representations for fractions. Thus, decimals appear to be much more effective than fractions in conveying information about magnitudes. This line of research suggests that fractions are not always interpreted as magnitudes, or at least that magnitudes are less easily derived from fractions than from decimals (perhaps because the two-dimensional format of fractions does not match the base-10 system associated with whole numbers).

Acquisition of Discrete and Continuous Quantities

A key conceptual distinction, which we will argue is intimately related to alternative notations for rational numbers, is the distinction between countable and continuous quantities, which align with integers and real numbers, respectively. This distinction has been highlighted in developmental research (e.g., Cordes & Gelman, 2005). Counting is appropriate when the relevant entities are discrete objects (e.g., the number of girls in a group of children), whereas measurement is appropriate for continuous mass quantities (e.g., height of water in a beaker). Continuous quantities can be subdivided into equal-sized units (i.e., discretized) to render them measurable by counting (e.g., slices of pizza), but the divisions are arbitrary in the sense that they do not isolate conceptual parts. Even for adults, the distinction between continuous and discrete quantities has a strong impact on transfer of mathematical procedures (Bassok & Holyoak, 1989; Bassok & Olseth, 1995). Procedures learned using discrete concepts transfer fairly readily to continuous concepts (which can be discretized); however, procedures learned using continuous quantities are almost impossible to transfer to discrete concepts (because there is no sensible way to "fill gaps" between discrete elements to create a meaningful continuous entity).

Research suggests that appreciation of continuous magnitudes is an early developmental achievement. Prior to learning how to count, infants and young children are already able to distinguish the magnitudes of different continuous quantities (Clearfield & Mix, 2001; Feigenson, Carey, & Spelke, 2002). This ability to make distinctions between continuous quantities reflects an approximate number sense, which is evolutionarily more primitive than exact calculations (Feigenson, Dehaene, & Spelke, 2004;

[1] Throughout this article we will refer to common fractions (i.e., numbers formed by a ratio of integers) simply as "fractions."

Dehaene, 1997). However, school-age children have an advantage when performing operations (e.g., counting) with discrete quantities (Gelman, 1993) over performing measurement operations using matched continuous quantities (Nunes, Light, & Mason, 1993). By this age, when students have learned to count, they can use numbers to achieve greater precision by counting than by estimation of continuous quantities.

The accuracy advantage afforded by exact computation strategies depends on acquiring competence in the necessary computation. Because exact computation is acquired later than the more elementary approximation strategies, children still acquiring competence in calculation may actually make *less* accurate judgments when quantities are discretized rather than continuous. For example, Boyer, Levine, and Huttenlocher (2008) found that when children were given a continuous quantity demarcated with lines to show equal parts, they were more likely to select an incorrect proportional match than when they made proportional judgments between two continuous quantities. Despite the fact that continuous judgments could have been made in either condition (by simply ignoring the lines in the "discrete" displays), discretization apparently triggered (imperfect) use of an exact calculation strategy, instead of the more primitive (but for novices, more accurate) approximation strategy. This finding provides evidence that from a young age, the ontological concept of discreteness serves as a strong cue for use of an exact calculation strategy.

Relational Structure of Rational Numbers

In this study, we focus on adult understanding of rational numbers in relation to discrete and continuous quantities. The core definition of a rational number—one that can be expressed as the quotient of two integers, *a/b*, where $b \neq 0$—specifies a relation between the cardinality of two sets. As Kieren (1976) has argued, a key conceptual function of rational numbers is to represent a variety of mathematical relations, such as part-whole, quotient, ratio number, operator, and measure. As alternative notations for rational numbers, fractions and decimals have typically been viewed as simply equivalent in magnitude, other than rounding error (e.g., 3/8 vs. 0.375). For example, the Common Core State Standards Initiative (2014) for Grade 4 refers to decimals as a "notation for fractions."[2] Research has shown that decimals are more effective than fractions for representing one-dimensional values of magnitude (DeWolf et al., 2014). However, because of their bipartite structure, factions seem to be better suited (relative to decimals) for representing relations between two distinct sets. A fraction represents the ratio formed between the cardinalities of two sets, each expressed as an integer. Because both the numerator and the denominator are free to vary, a fraction is inherently a two-dimensional structure (English & Halford, 1995; Halford et al., 1998, 2010). Although fractions do express magnitudes, they first and foremost express relations between countable entities. The inherent relationality of fractions may make them an important precursor to algebra, consistent with the importance of performance with fractions as a predictor of subsequent success in mathematics (Siegler et al., 2011, 2012). Figure 1 illustrates a relational structure, in the form $a/b = c$, that connects rational numbers to their magnitudes. Note that fractions and decimals play distinct roles within this structure. Specifically, the *a/b* component, which expresses a ratio between integers, has the form of a fraction; in contrast, the output of the implied division, *c*, expresses the magnitude of that relation, and can be expressed using the one-dimensional decimal notation (with magnitude less than 1 when $a <$ b).

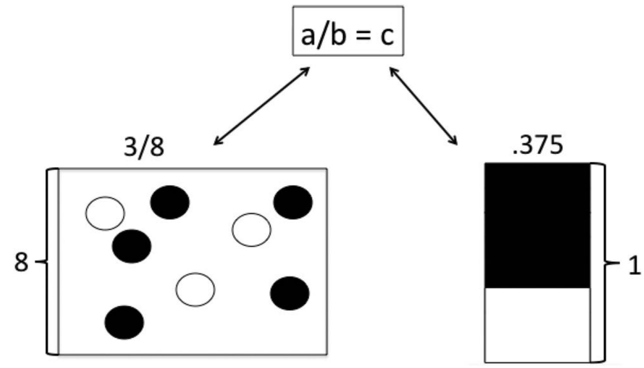

Figure 1. Modeling relations based on discrete or continuous quantities with fractions or decimals.

Importantly, though a fraction and its corresponding decimal convey (to some arbitrarily close approximation) the same magnitude, their distinct roles in the relational model give rise to meaningful conceptual distinctions. The displays shown in Figure 1 involve two subsets that are either broken down into countable units (left) or shown together as one continuous mass (right). Both fractions and decimals can serve as models of such visual displays, but have different natural alignments. The bipartite structure of a fraction provides a direct mapping to countable subsets in a visual display (e.g., a subset of 3 white objects within a set of 8 white and black objects; see Figure 1, left).

In contrast, a decimal is inherently unidimensional because the implied denominator is fixed (base 10) rather than variable (Halford et al., 1998, 2010). Accordingly, the decimal, relative to the fraction, yields a poorer conceptual match to the two-dimensional set structure of the discrete display. However, the decimal may provide a one-dimensional measure of a portion of a continuous unit, and hence is a better conceptual match to a visual display involving continuous masses (see Figure 1, right). Such continuous masses do not correspond to sets (without the additional step of dividing each mass into equal-sized discrete elements, thereby creating measurement units). Without first imposing such a measurement system on the continuous display, there is no basis for counting elements, and therefore there is no unique integer that characterizes the cardinality of each of the masses. Thus, the

[2] At a deeper level of analysis, the two number types are not representationally equivalent. Fractions represent rational numbers, whereas unbounded decimals represent the real numbers, of which the rational numbers are a subset. If decimals are bounded, they cannot exactly capture the magnitude of all real (or rational) numbers (e.g., 1/3); if unbounded, they also capture the irrational numbers (e.g., π), which have no exact fraction equivalent. Experimental work has for obvious reasons only used bounded decimals, the magnitudes of which at least closely approximate the magnitudes of matched fractions. However, the fundamental definition of common fractions (but not decimals) in terms of the cardinality of sets demonstrates that the two formats represent conceptually distinct number types, rather than simply alternative notations or typographical conventions.

continuous display provides a poor conceptual match to the fraction notation. In contrast, because the decimal notation inherently represents a one-dimensional quantity conveyed by the relative magnitudes of the two masses, it yields a better conceptual match than does the fraction.

These conceptual differences between fractions and decimals are intimately linked to differences in the mathematical procedures they afford. Table 1 summarizes the natural correspondences between fractions and decimals, respectively, with concepts and procedures. The experiments we report are all based on the 2 × 2 design outlined in Table 1, formed by the factorial combination of symbolic notation (fraction vs. decimal) and display type (discrete vs. continuous). Our central hypothesis is that fractions are conceptually linked to discrete sets and decimals to continuous masses. Because small sets of discrete elements can be counted, the preferred procedure for evaluating fractions relies on counting elements in sets. Thus, when fractions are applied to discrete displays, counting is likely to be evoked. Moreover, in many reasoning tasks, the solution can be achieved without converting the two-dimensional fraction into a one-dimensional magnitude. For example, to determine that the discrete display in Figure 1 (left) conveys the relation 3/8, it suffices to count the elements of each set and form the resulting fraction, which matches that given; there is no need to execute the further step of evaluating the magnitude of 3/8 (e.g., by applying division). In contrast, evaluating the fraction with respect to the continuous display (right) would require imposing (perhaps by mental "slicing") a measurement system on the masses to create equal-sized discrete units, which can then be counted (the "backup procedure"; see Table 1).

In contrast to a fraction, a one-dimensional decimal does not naturally align with the two components that form a ratio, but rather with the continuous quantity corresponding to its integrated magnitude (the value of a proportion). Hence the preferred procedure for evaluating a decimal as a match to a partitioned perceptual display would be estimation of a relative magnitude. People (and perhaps other animals) appear to be able to estimate proportions based on their system for approximate magnitude (Jacob, Vallentin, & Nieder, 2012; Nieder & Dehaene, 2009; Halberda & Feigenson, 2008). However, by their very nature, approximate magnitudes will be more error-prone than exact calculations (assuming the reasoner is competent in counting). Thus, decimals will tend to evoke estimation procedures for both discrete and continuous displays, leading to reduced accuracy relative to fractions with discrete displays that afford use of a counting procedure.

The "backup" procedure for decimals would be to use a variant of a counting strategy (see Table 1). Counting is possible for a discrete display; however, it would directly create a fraction, which would then have to be converted to a decimal by division. The extra division step would create difficulty in evaluating the decimal. For a continuous display, it would be necessary to first impose a measurement scale on the continuous masses, a process likely to be cognitively demanding. Thus, the superior alignment of the conceptual structure of a decimal to the continuous display may not translate into greater accuracy or speed in evaluating a match between the value of the decimal and of the depicted relative magnitude.

We will test the hypothesized differences between reasoning with fractions versus decimals in experiments using simple visual stimuli; however, there is reason to believe that the two-dimensional structure of fractions may also be the basis for other reasoning advantages associated with "natural frequency" formats (Gigerenzer & Hoffrage, 1995; Hoffrage, Gigerenzer, Krauss, & Martignon, 2002; Tversky & Kahneman, 1983). In the terminology of the present study, a natural frequency is a specific type of fraction format. We will consider the implications of our findings for understanding natural frequencies in the General Discussion.

Overview of Experiments on Reasoning With Rational Numbers

Our analysis of fractions as relational models leads to the prediction that, even though decimals are more effective in conveying one-dimensional magnitudes than are fractions (DeWolf et al., 2014), fractions should allow more accurate reasoning about bipartite relational structures, such as ratios defined on components of discrete perceptual displays. To test this basic hypothesis, we created pictorial displays of a set comprised of two subsets, paired with either a fraction or decimal value representing a certain ratio relation within the display (see Figure 2). The quantities in each display were either discrete, continuous, or continuous but divided into equal units (i.e., discretized). Experiment 1 tested the hypothesis that college-educated adults will, in fact, exhibit a preferential alignment between fractions and sets of discrete quantities, and between decimals and portions of continuous quantities. Importantly, we employed a task involving no computation, thus ensuring that the decision would be purely based on a conceptual preference for particular symbolic notations of rational numbers, rather than on the relative ease of performing computational procedures using these notations.

The subsequent experiments involved tasks that do require computational procedures to reason about relations. We focused on two types of ratio relations, both of which can be mapped to the structure of a fraction. A part-to-part ratio (PPR) is the relation between the size of the two subsets of a whole, whereas a part-to-whole ratio (PWR) is the relation between the size of one subset

Table 1
Correspondences Between Notations for Rational Numbers and Concepts/Procedures

Preferred conceptual quantity type	Symbolic notation	Preferred procedure	Back-up procedure
Discrete sets	Fraction	Count	*Continuous masses:* impose measurement scale, then count units
Continuous masses	Decimal	Estimate relative magnitude (proportion)	*Discrete sets:* count and divide *continuous masses:* impose measurement scale, then count units, then divide

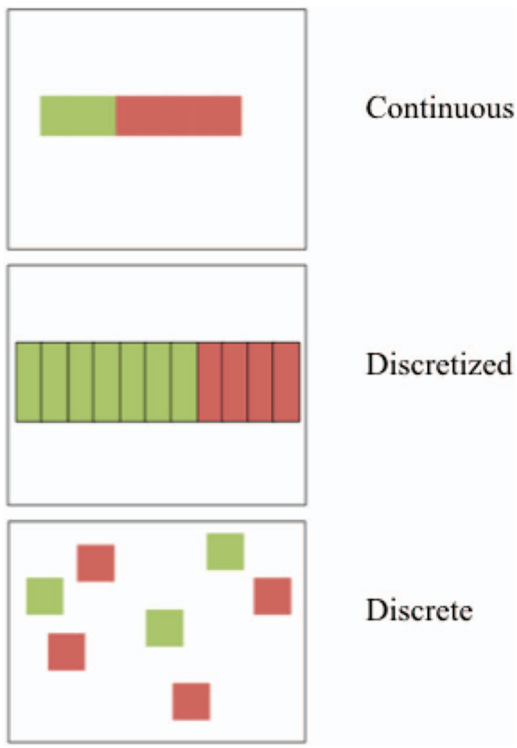

Figure 2. Examples of continuous, discretized, and discrete displays used in experiments. See the online article for the color version of this figure.

and the whole.[3] These two types of ratios were chosen in part because they offered a methodological advantage for administering a forced-choice task (Experiments 2–4). However, there are also important conceptual differences between these two types of relationships. The PWR is a conventional relationship for representing continuous magnitude. Creating an equivalent decimal for a PWR fraction is conceptually straightforward, and simply indicates a one-dimensional magnitude. By contrast, the PPR more directly reflects the abstract relational nature of a fraction. The decimal equivalent of a PPR is more difficult to understand in isolation, whereas a fraction highlights the comparison of interest (e.g., 2/3 might indicate 2 boys for every 3 girls, whereas .67 would indicate a .67 boy for every 1 girl). We included both types of relationships in order to evaluate the generality of fractions as notations for expressing relations.

The framework we propose (see Table 1) predicts that the two types of rational numbers are likely to evoke different procedures for evaluating such relations given different types of displays. When discrete units are provided, counting is a likely strategy, which (at least for well-educated students solving problems involving relatively small values) should generate accurate measures of subsets that align directly with the numerator and denominator of a fraction. For decimals, in contrast, counting of subsets/sets would require additional processing (e.g., mental division) to translate the result into decimal form. Alternatively, decimal magnitudes may be estimated directly (Jacob et al., 2012). However, estimation is likely to be less accurate than counting, resulting in more errors when decimals, rather than fractions, are paired with displays of countable items. For continuous displays, the bipartite format of fractions may still encourage counting (e.g., by mentally slicing the display into units). However, accuracy is likely to be sacrificed, reducing or eliminating the advantage of fractions over decimals.

Experiment 2 tested the hypothesis that adults will be more accurate in identifying ratio-type relations (presented in displays similar to those illustrated in Figure 1) when using fractions rather than decimals, as long as the quantities are discrete or discretized (i.e., countable). In Experiments 3 and 4, we extended this paradigm to examine the impact of the two types of rational numbers in a task that requires higher-order analogical reasoning. Based on the hypothesized role of conceptual alignment in mathematical modeling, we predicted that for displays of countable entities, fractions would yield greater accuracy than decimals in both relation identification and analogical reasoning.

Experiment 1

In Experiment 1 we examined whether adults, in fact, show consistent preferences for particular types of symbolic notations depending on the type of entities being represented. In particular, we tested the hypothesis that adults prefer to use fractions to represent countable ratio relationships and decimals to represent magnitudes of continuous quantities, even when no computation is required.

Method

Participants

Participants were 48 undergraduates at the University of California, Los Angeles (UCLA; mean age: 20.4 years; 37 females), randomly assigned in equal numbers to two between-subjects conditions (part-to-part vs. part-to-whole ratio; see below). All participants received course credit.

Materials and Design

The study was a 2 (relation type: part-to-part vs. part-to-whole ratios) × 3 (display type: continuous, discretized, discrete) design. As noted earlier, a part-to-part ratio (PPR) is the relation between the size of the two subsets of a whole, whereas a part-to-whole ratio (PWR) is the relation between the size of one subset and the whole. In our subsequent experiments, using two different relation types allowed us to create tasks involving a two-alternative forced choice. In Experiment 1 relation type was a between-subjects factor, and display type was a within-subjects factor.

Figure 2 depicts examples of the three display types. The discrete items were displays of circles, squares, stars, crosses, trapezoids, and cloud-like shapes. The continuous items were displays of rectangles that could differ in width, height, and orientation

[3] The term "fraction" is most commonly applied to part-whole or subset-set relations, which we term *part-to-whole* ratios. The term "ratio" is more commonly used for subset-subset relations. In the present study we refer to the latter as *part-to-part* ratios, and use the term "fraction" to embrace both. Similarly, we use the standard fraction notation (*a/b*) to represent ratios (more commonly notated *a:b*). In short, we use the familiar term "fraction" and its notation to cover a variety of bipartite structures based on a division relation.

(vertical or horizontal). The discretized items were identical to the continuous displays except that the rectangles were divided into equal-sized units by dark lines. For the stimuli used in test trials, red and green were used to demarcate the two different subsets (in practice trials, yellow and blue colors were used).

The displays varied which color represented the larger subset versus the smaller subset. Displays in all experiments were sized at approximately 700 pixels (width) × 800 pixels (height), viewed from a distance of about 50 cm.

Participants were given instructions for either the part-to-part ratio (PPR) or part-to-whole ratios (PWR) condition. They were given the following instructions for the PPR condition: "In this experiment, you will see displays that show various part-to-part relations. In the display below [display with 1 orange circle and 2 blue crosses] this would be the number of orange circles relative to the number of blue crosses. Such relations can be represented with fractions (e.g., 3/4) or with decimals (e.g., .75). For a complete list of the ratios used to create the displays, see Appendix Table A1. For each display your task is to choose which **notation** is a better representation of the depicted relation—a fraction or a decimal. Note that the specific values (i.e., 3/4 and .75) are just examples and do not match the values in the displays." For the PWR condition, the instructions were identical except for the description of the relations. In this condition the part-to-whole relation was defined using the example of the number of orange circles relative to the total number of blue crosses and orange circles. The relation type (PPR vs. PWR) was manipulated between subjects; thus participants in the PPR condition were only told about PPRs and participants in the PWR condition were only told about PWRs. Participants were shown examples of the continuous and discretized displays, in addition to the discrete display, and were told that displays could appear in any of those formats.

The task was simply to decide whether the relationship should be represented with a fraction or a decimal. Figure 3 shows an example of a discretized trial. In order to assess this preference on a conceptual level, the specific fraction and decimal shown to participants (3/4 and .75) were held constant across all trials, and never matched the number of items in the pictures. Thus (unlike the subsequent experiments we report), no mathematical task needed to be performed. There was therefore no requirement for accuracy, nor was any speed pressure imposed. Since the quantity shown in a display never matched the particular fraction and decimal values provided as response options, there was no real need to even determine the specific value represented in a display. The paradigm of Experiment 1 was thus intended to investigate participants' conceptual representations for fractions and decimals, in a situation in which mathematical procedures were not required.

Procedure

Stimuli were displayed with Macintosh computers using Superlab 4.5 (Cedrus Corp., 2004), and participant responses were recorded. Participants were given the instructions described above for either the PPR condition or the PWR condition. Because the choice values provided did not match the values depicted in the picture (they were always .75 for decimal and 3/4 for fraction), there was no accuracy measure. Also, there was no speed pressure to respond. Participants were told to select the "z" key for decimals and the "m" key for fractions. Participants completed 60 test trials (20 for each display type). A fixation cross was displayed for 600 ms between each trial. Display types were shown in a different random order for every participant.

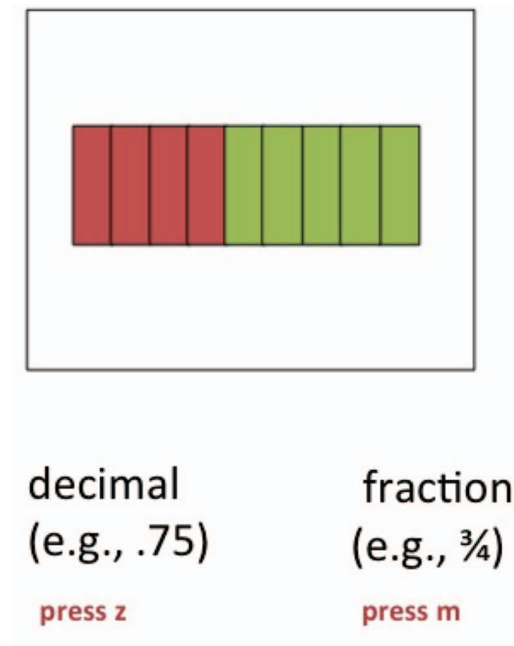

Figure 3. An example of a discretized trial from Experiment 1. Participants were shown the same decimal (.75) and fraction (3/4) as alternatives on every trial. See the online article for the color version of this figure.

Results and Discussion

Because participants were forced to choose either a fraction or a decimal for each trial, the preference for each is complementary. For simplicity, we report the preference for fractions. The proportion of trials in which participants selected the fraction notation was computed for each display type for each participant. Figure 4 shows the proportion of trials that participants picked either fractions or decimals for each display. A 2 (relation type: PPR vs. PWR) × 3 (display type: discrete, discretized, continuous) analysis of variance (ANOVA) was used to assess differences in notation preference. There was no interaction between relation type and display type, $F(2, 45) = .53$, $p = .59$; $\eta_p^2 = .02$, and there was no significant difference between PPR and PWR conditions, .57 versus, .62, $F(1, 46) = 1.74$, $p = .19$, $\eta_p^2 = .04$. However, there was a significant main effect of display type, $F(2, 45) = 23.33$, $p < .001$, $\eta_p^2 = .31$.

Planned comparisons showed that there was no overall difference between discrete and discretized displays, .68 versus .77; $F(1, 46) = 2.11$, $p = .15$, $\eta_p^2 = .04$. However, the preference for fractions was significantly lower for continuous displays than for either the discrete, .33 versus .68; $F(1, 46) = 15.24$, $p < .001$, $\eta_p^2 = .25$, or discretized displays, .33 versus .77; $F(1, 46) = 45.59$, $p < .001$, $\eta_p^2 = .50$. Thus, participants showed a strong preference for fractions with discrete and discretized displays, and a symmetrical preference for decimals with continuous displays.

The results of Experiment 1 revealed that adults prefer to represent both PPR and PWR ratio relationships with fractions

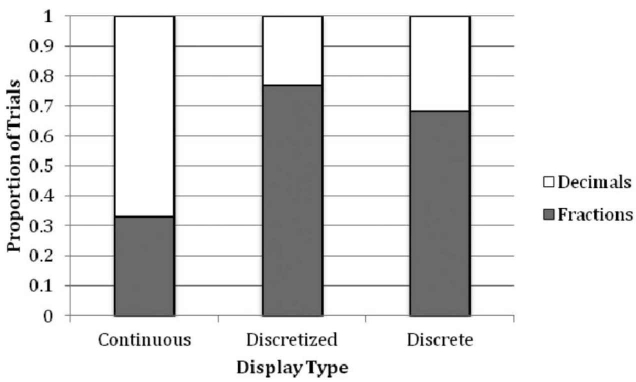

Figure 4. Proportion of trials for each display type in which either a fraction or decimal were chosen (Experiment 1).

when a display shows a partition of countable entities, but with decimals when the display shows a partition of continuous mass quantities. Participants picked the number format that provided the best conceptual match to either continuous or discrete displays, even though no procedural computation was required to complete the task. No mathematical task needed to be performed, and the specific quantities depicted in the displays did not match the numerical values of the fractions and decimals provided as choice options; hence, our findings demonstrate that the preferential association of display types (discrete or continuous) and rational number formats (fractions or decimals) has a conceptual basis. The results of Experiment 1 closely align with evidence that college-educated adults show a preference for using continuous displays to represent decimals and countable displays to represent fractions (Rapp, Bassok, DeWolf, & Holyoak, in press).

Experiment 1 thus provides strong support for the hypothesis that the natural alignment of different symbolic notations with different quantity types has a conceptual basis. Experiments 2–4 tested whether these conceptual alignments also hold for more complex tasks that require computations and procedures.

Experiment 2

Experiment 1 established a conceptual correspondence between quantity types and symbolic notations for rational numbers. Experiments 2–4 examined whether this conceptual correspondence also makes one or the other symbolic notation more effective in a relational reasoning task. Experiment 2 tested the hypothesis that adults would be better able to identify and evaluate ratio relationships using fractions than decimals, especially for discrete (or discretized) quantities.

Method

Participants

Participants were 58 UCLA undergraduates (mean age: 20.4 years; 49 females), randomly assigned in equal numbers to the two between-subjects conditions. Course credit was provided to participants.

Materials and Design

The study was a 2 (symbolic notation: fractions vs. decimals) × 2 (relation type: part-to-part vs. part-to-whole ratios) × 3 (display type: continuous, discretized, discrete) design. Although the distinction between the two types of relations is of potential theoretical interest in its own right, our main reason for using the two types was methodological, since it enabled us to create a two-alternative forced-choice task. Symbolic notation was a between-subjects factor, and relation type and display type were within-subjects factors.

The displays were similar to those used in Experiment 1 (see Figure 2). The magnitudes of fractions and decimals were matched. The values of the fractions and decimals were always less than one, and decimals were shown rounded to two decimal places. The values of the rational number presented on each trial represented one of two ratio relationships within the display: part-to-whole ratio (PWR) or part-to-part ratio (PPR). These were the same relationships used in Experiment 1, but the task in Experiment 2 explicitly required participants to identify on each trial which of the two relationships matched a presented number. Thus, a number was paired with the display that specifically matched one of the relationships.

For example, Figure 5 shows an example of a PWR trial with a display with 9 red units out of a total of 10. The number specified is 9/10 (or .90 in a matched problem using decimals), thus corresponding to a PWR. For the corresponding PPR problem, the number would be 1/9 (or .11 in decimal notation). The smaller subset would be the numerator in this case, so that the overall magnitude was always less than one. For a complete list of the stimuli used, see Appendix Table A2.

Procedure

Stimuli were displayed with Macintosh computers using Superlab 4.5 (Cedrus Corp., 2004), and response times and accuracy were recorded. Participants received the following instructions: "In this experiment, you will see a display paired with a value. You need to identify which of the two following relationships is shown." Below this, there were two different displays showing the PWR and PPR relations, which were simply referred to as "Relation 1" and "Relation 2." The assignment of the labels was counterbalanced for all subjects such that half were told Relation 1 was PPR and the other half were told Relation 1 was PWR. The PPR display contained 1 circle and 2 crosses. For the fractions condition

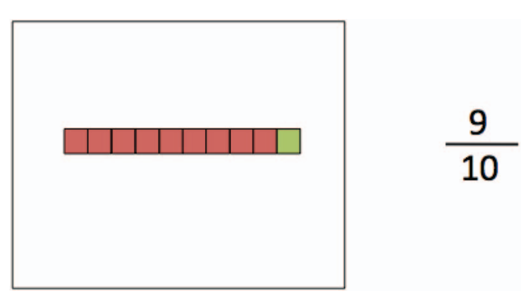

Figure 5. Example of a ratio identification problem (Experiment 2). See the online article for the color version of this figure.

this was labeled as “1/2 amount of circles per amount of crosses”; for the decimals condition it was labeled as “.50 amount of circles per amount of crosses.” The PWR was represented by a display of 2 circles and 3 crosses. For the fractions condition this was labeled as “2/5 of the total is the amount of circles”; for the decimal condition it was labeled as “.40 of the total is the amount of circles.” The first of these explanations of the PPR and PWR relations was shown with discrete items.

The subsequent screen showed the same values paired with discretized displays. A third screen showed the same values paired with continuous displays. Half of the participants were told to select the “z” key for Relation 1 and to select the “m” key for Relation 2; the other half received the reverse key assignments.

After this introduction, participants were given an example problem and asked to identify the relation. After they made their judgment, an explanation was shown to participants about why the example showed the correct relation. The explanation also stated what the numerical value would be for the problem if it had shown the alternative relation. Participants were then given another example using the other relation, with the same explanation process. A series of practice trials was then administered. Participants had to complete at least 24 practice trials (four for each of the six within-subjects conditions). If they scored at least 17 correct (i.e., about 70%) they were able to move on to the test trials. If they did not score above this threshold, they continued with additional practice trials until they reached the threshold percent correct. About 75% of participants passed the threshold after the first round of practice trials. The remaining participants were able to advance after a second set of practice trials. All of the practice trials were different from those used in the test trials. Feedback was given for incorrect trials, in the form of a red “X” on the screen. After the practice trials had been completed, a screen was displayed informing participants that the actual test trials were beginning. Participants were told to try to go as quickly as possible without sacrificing accuracy. They completed 72 test trials (12 for each of the 6 within subjects conditions). A fixation cross was shown for 500 ms between each trial. Feedback was continued for incorrect trials. Relation types and display types were shown in a different random order for every participant.

Results and Discussion

Accuracy and mean response time (reaction time (RT)) on correct trials were computed for each condition for each participant. A 3 (display: continuous, discretized, discrete) × 2 (relation type: PPR vs. PWR) × 2 (symbolic notation: fraction vs. decimal) mixed factors ANOVA was used to assess differences in RT and accuracy. Since the three-way interaction was not reliable, $F(2, 112) = .97$, $p = .38$, $\eta_p^2 = .02$,, all analyses are reported after collapsing across the factor of relation type.

Figure 6 displays the pattern of accuracy, the primary dependent measure in Experiments 2–4. Accuracy exceeded chance level (50%) for all conditions. For some conditions, accuracy was lower than the practice threshold level, likely because the practice trials comprised a different set of problems that were less challenging than the actual trials. There was a significant interaction between display type and symbolic notation, $F(2, 55) = 24.57$, $MSE = 1.7$, $p < .001$. $\eta_p^2 = .47$. Planned comparisons revealed that participants were more accurate when using fractions than decimals for discrete displays, 81% versus 63%; $F(1, 56) = 23.64$, $MSE = 8.2$, $p < .001$ $\eta_p^2 = .30$, and discretized displays, 78% versus 63%; $F(1, 56) = 13.92$, $MSE = 9.7$, $p < .001$, $\eta_p^2 = .20$. In contrast, accuracy did not differ as a function of symbolic notation for continuous displays, 62% versus 67%; $F(1, 56) = 1.52$, $MSE = 10.4$, $p = .22$, $\eta_p^2 = .03$. There was a significant main effect of relation type favoring PPR over PWR, 72% versus 66%, $F(1, 56) = 11.00$, $MSE = .024$, $p = .002$, $\eta_p^2 = .16$; however, there was no interaction between relation type and number type, $F(1, 56) = .03$, $MSE = .024$, $p = .87$, $\eta_p^2 < .001$, nor an interaction between relation type and display type, $F(2, 112) = 1.79$, $MSE = .02$, $p = .17$, $\eta_p^2 = .03$. Thus, participants across all conditions performed better on the PPR problems than the PWR problems (perhaps because PWR problems require an extra computational step to add the two subsets to find the whole).

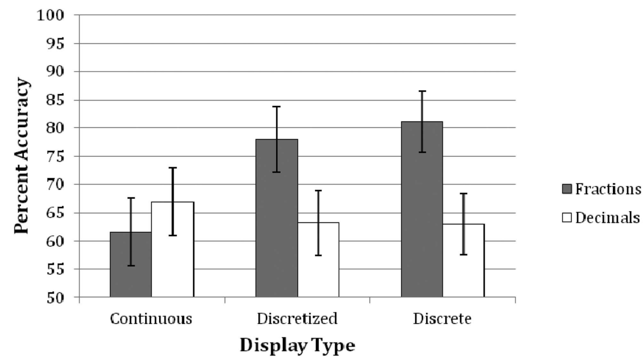

Figure 6. Accuracy of relation identification using fractions and decimals across different types of displays (Experiment 2). Error bars indicate standard error of the mean.

We also analyzed RTs primarily to assess any possible speed–accuracy trade-offs. Figure 7 displays the pattern of mean correct RTs across conditions. As found for the accuracy analysis, a significant interaction was obtained between display type and symbolic notation, $F(2, 55) = 3.52$, $MSE = 1369$, $p = .037$, $\eta_p^2 = .11$. Although RTs tended to be faster for fractions than decimals for discrete displays, the difference was not reliable, 4.20 s versus 5.97 s; $F(1, 56) = 2.40$, $MSE = 7607$, $p = .13$, $\eta_p^2 = .04$. A nonsignificant trend was also obtained for discretized displays, 3.93 s versus 4.59 s; $F(1, 56) = 0.585$, $MSE = 4357$ $p = .45$, $\eta_p^2 = .01$. These RT analyses confirm that the accuracy advantage of fractions over decimals for countable displays is not attributable to speed–accuracy trade-offs. For continuous displays, however, response times were significantly slower with fractions than with decimals, 3.77 s versus 2.88 s; $F(1, 56) = 4.77$, $MSE = 9202$, $p = .03$, $\eta_p^2 = .08$.

As in the accuracy analyses, there was also a significant main effect of relation type such that PPR problems were solved more quickly than PWR problems, 4.07 s versus 4.37 s; $F(1, 56) = 8.45$, $MSE = 9455$, $p = .005$, $\eta_p^2 = .13$. However, there was no significant interaction between relation type and number type, $F(1, 56) = .20$, $MSE = 9455$, $p = .67$, $\eta_p^2 = .004$, nor between relation type and display type, $F(2, 112) = .65$, $MSE = 13693$, $p = .52$, $\eta_p^2 = .01$.

In order to evaluate whether participants preferentially engaged in counting strategies when evaluating fractions rather than deci-

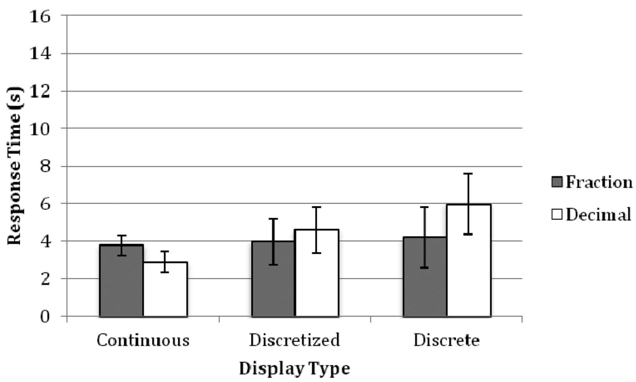

Figure 7. Mean response time for relation identification using fractions and decimals across different types of displays (Experiment 2). Error bars indicate standard error of the mean.

mals, we conducted regression analyses on the averaged data for each individual trial. The main variable of interest was the size of the denominator for the correct ratio. If a counting strategy were used, then time to identify the correct ratio would be expected to increase with the size of the larger subset (i.e., the subset corresponding to the denominator of the ratio). For each symbolic notation, analyses were performed separately for accuracy and for response times on correct trials, for each display type. To increase the sensitivity of the analyses, we included ratio type as a covariate, and also the logarithm of the numerical difference between the correct ratio and the alternative ratio. Difference between alternatives was defined as the difference between the given value and the value that would be the result of the alternative ratio. This factor was likely to predict problem difficulty, since previous research has shown that it is more difficult for adults to compare the relative sizes of two magnitudes (expressed as fractions or ratios) as the difference between the magnitudes decreases (DeWolf et al., 2014; Schneider & Siegler, 2010). Thus, we investigated whether evidence of a counting strategy would emerge after controlling for other likely sources of differences in problem difficulty. As no reliable effects of denominator size emerged for either symbolic notation in regression analyses based on accuracy, we only report analyses based on response times.

In accord with the hypothesis that fractions will be associated with use of counting, denominator size emerged as a significant predictor of RTs (after controlling for the two covariates) for each of the three display types: discrete, $b = .284$, $t(20) = 6.76$, $p < .001$, discretized, $b = .247$, $t(20) = 7.83$, $p < .001$, and continuous, $b = .16$, $t(20) = 4.56$, $p < .001$. The fact that denominator size was reliable even for the continuous displays suggests that participants who evaluated fractions used a counting-like strategy even when the display did not provide measurement units (perhaps by imagining such units using the backup procedure specified in Table 1). A very different pattern emerged when evaluations were based on decimals. Denominator size was not a reliable predictor of response times for either discrete, $b = .164$, $t(20) = .958$, $p = .35$, or continuous displays, $b = -.01$, $t(20) = .267$, $p = .79$, though it was marginally significant for discretized displays. $b = .285$, $t(20) = 2.09$, $p = .05$. The regression analyses on RTs thus supported the hypothesis that counting strategies tend to be used when evaluating fractions. Decimals, by contrast, may have been evaluated using an estimation procedure, which we hypothesize is preferred (see Table 1), and which would be more direct (and no less accurate) than a "count-and divide" strategy.

Overall, the results of Experiment 2 revealed an advantage for identifying ratio relationships in displays when these ratios were represented by fractions rather than decimals. However, this pattern was moderated by the nature of the visual displays. When displays conveyed countable entities (sets of discrete objects, or continuous displays parsed into units of measurement), ratios were evaluated more accurately when the symbolic notation was a fraction rather than a decimal. In contrast, when the display showed continuous quantities without measurement units, accuracy in evaluating ratio relations did not differ, and decisions were made more quickly for decimals than fractions.

Although the relation-identification task used in Experiment 2 yielded an interaction between number type and display type, its quantitative form (at least for the accuracy measure) differed from the similar interaction obtained in the purely conceptual task used in Experiment 1. Experiment 1 revealed a significant preference for decimals over fractions with continuous entities, but Experiment 2 did not find a clear accuracy advantage for decimals in any condition, even when paired with continuous displays (though the accuracy trend in the latter condition did favor decimals, and decimals yielded faster response times). The lack of a clear accuracy advantage for decimals paired with continuous displays in Experiment 2 likely reflects the important role that mathematical procedures play in relation identification (see Table 1). In particular, comparing the value of a decimal to a display requires either estimation or (for discrete or discretized displays) counting and division. As estimation is inherently more error-prone than counting strategies (at least for college students); this procedural requirement prevented decimals from showing a clear accuracy advantage over fractions even in the continuous condition. However, the estimation procedure apparently used for decimals in the continuous condition does seem to be faster to execute than the procedure used in this condition for fractions.

Overall, the findings of Experiment 2 highlight the importance of the relational structure internal to fractions. When the values within the fraction can be mapped to particular entities (i.e., when they are countable), identifying ratio relationships in visual displays is greatly facilitated. Decimals, in contrast, do not exhibit the same type of internal structure; hence, it is more difficult to map the integrated decimal value to two separate subsets within a display. However, when displays are continuous, the two subsets are difficult to measure exactly. Presumably, people are then forced to use more approximate estimate strategies, for which decimals yield about the same accuracy as fractions, with greater processing speed.

Experiment 3

Experiment 2 demonstrated an advantage of fractions over decimals in the identification of ratio relationships in displays with countable subsets and values. In Experiment 3, we investigated whether the fraction advantage (modulated by the nature of the visual displays) would extend to a more complex analogical reasoning task. To address this question, we extended the ratio-identification paradigm used in Experiment 2. In the analogy task introduced in Experiment 3, participants had to use the ratio relationship they identified in an initial source display to select an analogous value for the same type of ratio in a target display. Whereas the task used in Experiment 2 required identification of a first-order relation in a display (ratio between two quantities), the task used in Experiment 3 required computing a higher-order relation between the ratio relation extracted in the source display and the analogous relation in a target display. This reasoning task is more complex than that used in Experiment 2 because participants must not only identify the relationship between display and number, but also apply this relationship to a novel display and identify the correct value. If the internal relational structure of a fraction is an important aspect of its meaning, then the general pattern of performance differences observed in Experiment 2 should also be obtained in a task involving high-level analogical reasoning.

Method

Participants

Participants were 52 undergraduates at UCLA (mean age = 21; 30 females) who received course credit. They were randomly assigned in equal numbers to the two between-subjects conditions.

Materials and Design

Similar to Experiment 2, a 2 (symbolic notation: fractions vs. decimals) × 2 (relation type: part-to-part vs. part-to-whole ratios) × 3 (display type: continuous, discretized, discrete) design was employed. Symbolic notation was manipulated between subjects, whereas relation type and display type were manipulated within subjects. As in Experiment 2, discrete, discretized, and continuous displays were paired with either a fraction or decimal that represented a PPR or PWR. The analogy problems used in Experiment 3 (Figure 8) can be viewed as a generalization of the proportional analogy format A:B :: C:D versus D′, where A and C are ratios in the source and target displays, respectively, and C, D, and D′ are corresponding rational numbers. (Since ratios are themselves relations, these analogy problems actually involved a third-order relation.) The source analogs (A:B) were identical to the problems used in Experiment 2. Participants needed to identify which of two numbers in the target (D vs. D′) correctly matched the display with the same relationship specified in the source. The analogy task required making a choice of the correct number to complete the target analog using the same relation as in the source. The symbolic notation (fraction or decimal) was always the same across the source and target. Solving an analogy problem required first identifying the ratio relation in the A display characterized by the number given as B (as in Experiment 1). Once the higher-order relation between A and B was extracted, the solution required identifying the same relation type in target display C, and choosing the corresponding number as the D term. The D′ foil was always chosen to match the alternative ratio relation in the C display.

Since the analogy problems were constructed using the problems from Experiment 2 as the A:B source, the specifications of the stimuli are the same. The same two colors, green and red, were used in the A and C displays, and the color relationship was maintained, such that the same color mapped to the same part of the relation in both A and C. This constraint served to identify which part (lesser or greater) mapped to the numerator in a ratio relation. As in Experiment 2, color assignments varied across trials, so the same color might indicate the lesser subset on one trial and the greater subset on another. For each trial, the source and target were randomly assigned for each participant. Thus, the only aspect that was consistent between the two analogs was the higher-order relationship (PPR or PWR) and the color mapping. For a complete list of the source and target stimuli used, see Appendix Table A3.

Procedure

The procedure was identical to Experiment 2, except expanded to constitute an analogy task. Participants were given the following instructions: "In this experiment, there will be two different types of relations between the picture and the numerical value shown below." Below this instruction appeared the same two displays for the two types of ratios, as in Experiment 2 with the same labeling. Participants were then told: "The first step is to identify the relation between the top picture and numerical value (shown with an example of a source display as in the top of Figure 8). The second step is to select one of the two numerical values on the right that shares the same relationship with the bottom picture as the top picture" (shown with an example of a target display as in the bottom of Figure 8). For the first step, instead of hitting a key that corresponded to a specific relationship (as in Experiment 2), participants simply hit the space bar when they had identified the relationship. After the space bar was pressed, the target (C:D vs.

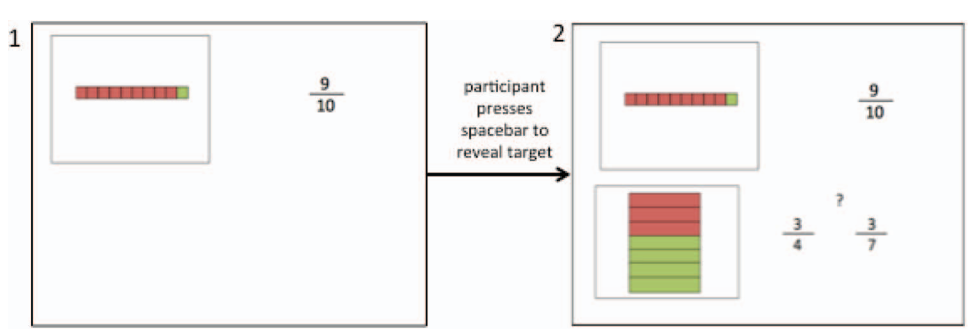

Figure 8. Example of an analogy trial used in Experiments 3 and 4. See the online article for the color version of this figure.

D′) was shown on the screen below the source, so that both the source and target were on the screen simultaneously. Participants were asked to select which of two numbers (D or D′) shared the same relationship with this display as the relationship from the source. Half of the time, D appeared on the right side of the screen. They made their selection by pressing the “z” key for the number shown on the left and the “m” key for the number shown on the right. The “z” and “m” keys were labeled with “L” and “R,” respectively, so that participants could remember which key corresponded to each number. As in Experiment 2, participants were told to try to go as quickly as possible without sacrificing accuracy. After reading the instructions and completing 12 practice trials with feedback, participants proceeded to the 72 test trials.

Results and Discussion

As in Experiment 2, accuracy (the primary dependent measure) and mean RT on correct trials were computed for each condition for each participant. RTs were measured from the onset of the source problem to the selection of the correct value for the target display. A mixed factors ANOVA was used to compare differences in RT and accuracy. No reliable overall differences were obtained between the two relation types (PPR and PWR) on either measure, accuracy: $F(1, 50) = 2.58$, $p = .11$, $\eta_p^2 = .05$; RT: $F(1, 50) = 0.73$, $p = .40$, $\eta_p^2 = .01$, so all results reported here are collapsed across this factor, as in Experiment 2.

As shown in Figure 9, the pattern of results for analogical accuracy largely followed the pattern observed in Experiment 2 for the simpler relation identification task. A significant interaction was obtained between display type and symbolic notation, $F(2, 49) = 20.59$, $MSE = 1.8$ $p < .001$, $\eta_p^2 = .46$. Planned comparisons indicated that accuracy was higher for fractions than decimals in the discrete condition, 87% versus 66%; $F(1, 50) = 28.96$, $MSE = 7.3$ $p < .001$, $\eta_p^2 = .37$, and discretized condition, 80% versus 67%; $F(1, 50) = 10.06$, $MSE = 8.3$, $p = .003$, $\eta_p^2 = .17$, but did not differ across the two symbolic notations for the continuous condition, 61% versus 65%; $F(1, 50) = 0.86$, $MSE = 7.7$, $p = .36$, $\eta_p^2 = .02$.

The accuracy analysis yielded a reliable 3-way interaction between relation type, display type, and number type, $F(2, 100) = 6.89$, $MSE = .02$, $p = .002$, $\eta_p^2 = .12$. This interaction was driven by a significant advantage in accuracy for PWR problems over PPR problems only for decimals paired with continuous displays, 74% versus 55%, $F(1, 50) = 23.76$, $MSE = .04$, $p < .001$, $\eta_p^2 = .32$. No such interaction was apparent in Experiment 2, and we have no explanation for this finding.

In analyzing RTs, the response times for incorrect answers were excluded, as were RT outliers that were greater than three standard deviations from the mean (roughly 2% of response times). As shown in Figure 10, the RT pattern is consistent with the pattern of accuracy results, and largely replicates the pattern observed in Experiment 2 (though RTs in Experiment 3 were of course, much longer, since they reflect the duration of the entire analogy problem, not just processing of the source). In particular, there was a reliable interaction between symbolic notation and display type, $F(2, 49) = 16.19$, $MSE = 2721$, $p < .001$, $\eta_p^2 = .40$. Planned comparisons indicated that RTs were faster with fractions than decimals for the discrete condition, 8.5 s versus 12.8 s; $F(1, 50) = 7.10$, $p = .01$, $\eta_p^2 = .12$, with a trend for the discretized condition, 8.3 s versus 11.2 s; $F(1, 50) = 3.51$, $p = .07$, $\eta_p^2 = .07$. For the continuous conditions, RTs for fractions versus decimals did not differ reliably, 9.3 s versus 7.7 s; $F(1, 50) = 2.12$, $p = .15$, $\eta_p^2 = .04$. The general pattern of RT differences was thus similar to that observed in the simpler relational verification task (Experiment 2), except that there was a general shift toward an RT advantage for fractions over decimals.

No reliable interaction was observed between relation type and number type, $F(1, 50) = .27$, $MSE = 3718$, $p = .61$, $\eta_p^2 = .01$, nor between relation type and display type, $F(2, 49) = 1.85$ $MSE = 2721$, $p = .16$, $\eta_p^2 = .03$. The 3-way interaction between the three factors was also not reliable, $F(2, 100) = 3.24$, $MSE = 2728$, $p = .05$, $\eta_p^2 = .06$.

As was the case for the ratio identification task used in Experiment 2, the analogy task used in Experiment 3 showed an overall advantage for solving problems using fractions as compared to decimals. In Experiment 3, participants not only had to correctly identify a particular relationship between a numerical value and a ratio relation in a display, but also had to use this relationship to identify which of two values correctly mapped to the same type of ratio relation in a new display (where the ratio quantity differed between source and target). Unlike Experiment 2, RTs in Experiment 3 showed a significant RT advantage for fractions over decimals in solving problems using discrete and discretized displays, whereas RTs for the two symbolic notations did not differ for continuous displays (rather than showing a reversal as in Experiment 2). Thus, if anything, the overall fraction advantage was yet more pronounced in the complex analogical reasoning task employed in Experiment 3. These findings imply that people are able to more quickly identify a number that correctly maps to a ratio relation when the symbolic notation of that number affords a one-to-one mapping to the conceptually relevant units within the displays, as is the case for a fraction. The fraction advantage extends beyond the identification of ratio relationships, since fractions also facilitate mapping of higher-order relations between types of ratios.

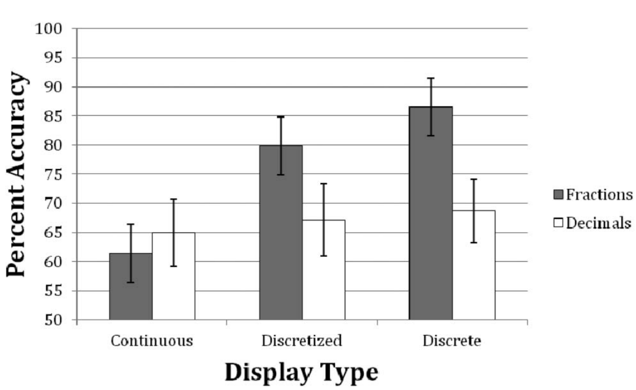

Figure 9. Accuracy of analogical inferences using fractions and decimals across different types of displays (Experiment 3). Error bars indicate standard error of the mean.

Experiment 4A

In the fraction conditions used in Experiments 2 and 3, the number of items in the display directly corresponded to the paired

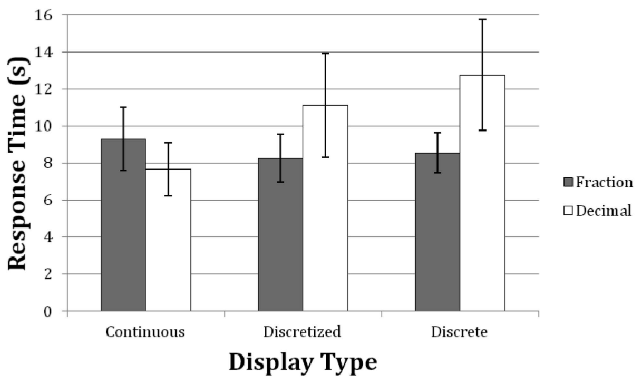

Figure 10. Mean response time for analogical inference using fractions and decimals across different types of displays (Experiment 3). Error bars indicate standard error of the mean.

fraction value. For example, if the number was 2/3 and the ratio relation was a PPR, then the display would display 2 items of one type and 3 items of another type. In the corresponding decimal condition, the number would appear as .67. However, there are many item arrangements that could fit this ratio (2/3, 4/6, 8/12, etc.). Experiment 4A was designed to determine whether the fraction advantage found in the analogy task used in Experiment 3 would still be obtained if the numbers in the numerator and denominator did not directly match the specific quantities shown in the display. Experiment 4A included fractions that were equivalent in overall value, but did not map one-to-one with the quantities in the displays. For example, a display with two items of one type and three items of another might be paired with 4/6, rather than 2/3.

Method

Participants

Participants were 75 UCLA undergraduates (mean age = 20.4; 53 females) who received course credit, randomly assigned in equal numbers to the three between-subjects conditions.

Materials and Design

Experiment 4A had one within-subjects factor (relation type) with 2 levels: PPR versus PWR, and one between-subjects factor (symbolic notation) with 3 levels: one-to-one (OTO) fractions, non-one-to-one (NOTO) fractions, and decimals. Because the findings of the previous experiments demonstrated that the largest advantage in accuracy for fractions over decimals is found for the discrete display type, only problems based on discrete displays were tested. All conditions included exactly the same set of displays. However, the OTO fractions condition used values that mapped one-to-one with the displays (e.g., if the relation was PWR with 3 items out of 7 items, the fraction would be 3/7). In contrast, NOTO fractions had numerators and denominators that were either two or three times greater or smaller than the actual number of items shown (e.g., 3 out of 7 items paired with 6/14). In the decimal condition, the same display was paired with the decimal equivalent (e.g., 3 out of 7 items paired with .43). There were 24 problems for each of the two relation conditions (PPR, PWR) for a total of 48 problems.

Procedure

The procedure was basically the same as in Experiment 3, except that only discrete displays were shown. Participants saw the same directions as in Experiment 3 (with only discrete examples and discrete practice questions). For a complete list of the stimuli used, see Appendix Table A4. They were told to try to go as quickly as possible without sacrificing accuracy. After completing 12 practice trials with feedback, participants continued on to the test trials and were given feedback throughout the task.

Results and Discussion

As in the previous experiments, accuracy and mean RTs on correct trials were computed for each condition for each participant. RTs were again measured from the onset of the source displays to the selection of the correct value for the target display. A mixed factors ANOVA was used to compare differences in RT and accuracy. Figure 11 shows the pattern of accuracy across conditions. There was a significant main effect of symbolic notation, $F(2, 72) = 27.79$, $MSE = 4.2$, $p < .001$, $\eta_p^2 = .44$, and relation type, $F(1, 72) = 8.33$, $MSE = 1.1$, $p = .005$, $\eta_p^2 = .11$. Planned comparisons showed that accuracy for OTO fractions was significantly higher than accuracy for decimals, 95% versus 65%, $F(1, 72) = 55.24$, $MSE = 8.4$, $p < .001$, $\eta_p^2 = .44$, or for NOTO fractions, 95% versus 78%, $F(1, 72) = 17.56$, $MSE = 8.4$, $p < .001$, $\eta_p^2 = .20$. Accuracy for NOTO fractions was significantly higher than accuracy for decimals, 78% versus 65%, $F(1, 72) = 10.02$, $MSE = 8.4$, $p = .002$, $\eta_p^2 = .12$.. Thus, fractions maintained an accuracy advantage over decimals for analogical reasoning with discrete displays, even when the numerator and denominator of the fraction do not equal the corresponding quantities in the visual display (NOTO fraction condition).

Unlike Experiment 3, a significant main effect of relation type was found in Experiment 4A, $F(1, 72) = 8.33$, $MSE = 1.1$, $p = .005$, $\eta_p^2 = .11$. Mean accuracy was 82% for PPR problems and 77% for PWR problems. In addition, a small but significant interaction was obtained between relation type and symbolic notation, $F(2, 72) = 3.18$, $MSE = 1.1$, $p = .047$, $\eta_p^2 = .08$. Since this

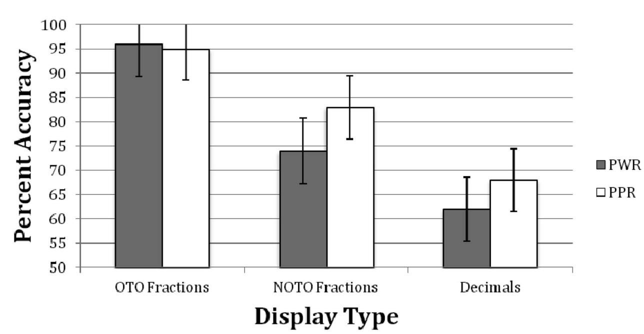

Figure 11. Accuracy of analogical inferences using fractions and decimals across different types of displays (Experiment 4A). Error bars indicate standard error of the mean.

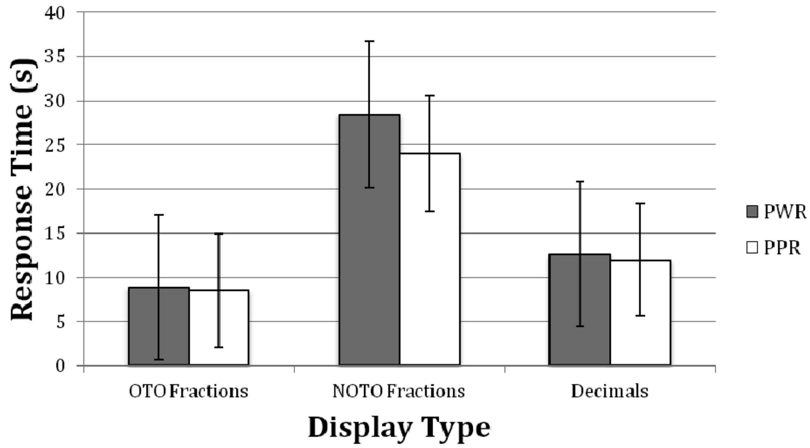

Figure 12. Mean response time for analogical inference using fractions and decimals across different types of displays (Experiment 4A). Error bars indicate standard error of the mean.

interaction did not replicate in Experiment 4B, we do not consider it further.

Figure 12 shows the corresponding pattern of response times. For RTs, the interaction between relation type and symbolic notation was not reliable, $F(2, 72) = 1.63$, $MSE = 4304$, $p = .20$, $\eta_p^2 = .04$, nor was the main effect of relation type, $F(1, 72) = 3.00$, $MSE = 4304$, $p = .09$, $\eta_p^2 = .04$, nor the main effect of symbolic notation, $F(1, 72) = 1.78$, $MSE = 2648757$, $p = .18$, $\eta_p^2 = .05$. There was no effect of relation type, PPR versus PWR: 15.2 s versus 17.0 s, $F(1, 71) = 3.00$, $MSE = 43004$, $p = .09$, $\eta_p^2 = .04$, and no interaction between relation type and number type, $F(2, 71) = 1.63$, $MSE = 43004$, $p = .20$, $\eta_p^2 = .04$. Clearly the considerable variance in RTs contributed to the lack of statistically reliable RT differences. Nonetheless, it is notable that the NOTO fractions condition yielded mean RTs considerably longer than those for either of the other two symbolic notations.

Experiment 4B

In Experiment 4A, response times for the NOTO fraction condition were considerably longer than those for the OTO fraction and decimal conditions. One explanation for this pattern is that the components of NOTO fractions are proportional to the relevant quantities in the displays. Even though extra time is required for a NOTO fraction because the depicted quantities do not equal the numerator and denominator, a counting strategy can yield higher accuracy than the approximate strategy associated with decimals. Nonetheless, an alternative possibility is that the accuracy advantage of the NOTO fraction condition over decimals in Experiment 4A was simply the consequence of a speed–accuracy trade-off. To evaluate the latter possibility, Experiment 4B used an identical design and procedure as Experiment 4A, except that the instructions were altered to encourage participants to take their time and try to achieve high accuracy (whereas in Experiment 4A, participants were told to go as fast as possible without sacrificing accuracy). If decimals are able to support analogical reasoning with ratios just as well as NOTO fractions, then the two conditions should not differ in accuracy in the absence of speed pressure.

Method

Participants were 66 UCLA undergraduates (mean age: 20.6; 53 females) who received course credit, randomly assigned in equal numbers to the three between-subjects conditions.

The design and procedure was identical to Experiment 4A with the exception that participants were directed to spend as much time as necessary on each problem to achieve high accuracy.

Results and Discussion

Figure 13 shows the pattern of accuracy across conditions. Consistent with the removal of time pressure, accuracy was generally higher in Experiment 4B than 4A. The analysis revealed a significant effect of symbolic notation, $F(2, 63) = 10.23$, $MSE = 5$, $p < .001$, $\eta_p^2 = .25$. Planned comparisons revealed that OTO fractions yielded higher accuracy than decimals, 94% versus 73%, $F(1, 63) = 20.30$, $MSE = 9.9$, $p < .001$, $\eta_p^2 = .24$. Moreover, NOTO fractions also yielded significantly higher accuracy than decimals, 85% versus 73%, $F(1, 63) = 6.802$, $MSE = 9.9$, $p = .01$, $\eta_p^2 = .10$. There was a trend toward greater accuracy for OTO fractions compared to NOTO fractions, 94% versus 85%, $F(1, 63) = 3.60$, $MSE = 9.9$, $p = .06$, $\eta_p^2 = .05$.

In contrast to Experiment 4A, the interaction between relation type and symbolic notation was not reliable in Experiment 4B, $F(2, 63) = .68$, $MSE = 1.2$, $p = .51$, $\eta_p^2 = .02$. However, there was again a significant effect of relation type, $F(1, 63) = 4.11$, $MSE = 1.2$, $p = .047$, $\eta_p^2 = .06$. Mean accuracy was 86% for PPR problems and 82% for PWR problems.

Figure 14 shows the pattern of RTs across conditions. As in Experiment 4A, there was no reliable interaction, $F(2, 63) = 2.30$, $MSE = 7983$, $p = .11$, $\eta_p^2 = .07$, or main effect for relation type, $F(1, 63) = 0.95$, $MSE = 7983$, $p = .33$, $\eta_p^2 = .02$; however, there was a significant effect of symbolic notation, $F(2, 63) = 7.59$, $MSE = 132138$, $p = .001$, η_p^2.19. Planned comparisons showed that there was no significant RT difference between NOTO fractions and decimals, 20.17 s versus 19.47 s, $F(1, 63) = 0.01$, $MSE = 264737$, $p = .91$, $\eta_p^2 = 0$. However, OTO fractions yielded faster RTs than NOTO fractions, 11.06 s versus 20.17 s (F(1,

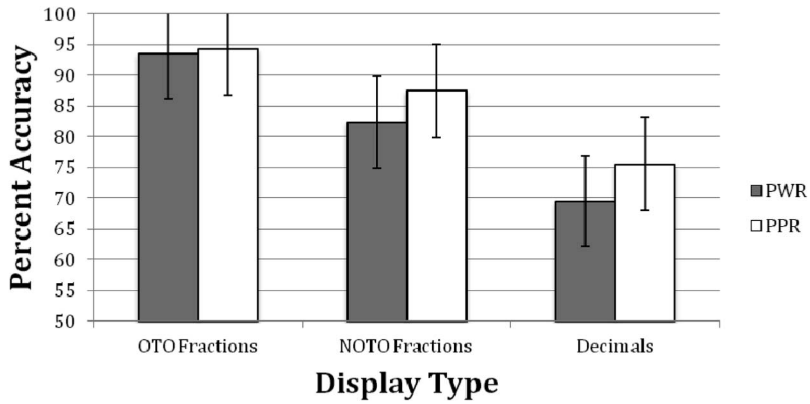

Figure 13. Accuracy of analogical inferences using fractions and decimals across different types of displays (Experiment 4B). Error bars indicate standard error of the mean.

63) = 11.02, *MSE* = 264737, p = .002, η_p^2 = .15, and also decimals, 11.06 s vs. 19.46 s, $F(1, 63) = 11.75$, *MSE* = 264737, p = .001, η_p^2 = .16. There was no reliable difference in RTs between PPR and PWR problems, 16.82 s versus 16.33 s, $F(1, 63) = .95$, *MSE* = 79832, η_p^2 = .02, and no interaction between relation type and number type, $F(2, 62) = 2.30$, $p = .11$, $\eta_p^2 = .07$.

The results of Experiment 4B demonstrated that even when participants solve analogy problems without any speed pressure, and RTs for the two critical conditions are quite closely matched, accuracy is higher for NOTO fractions than for decimals. Accuracy did not significantly differ between NOTO and OTO fractions, indicating that the fraction advantage is not dependent on whether they were reduced or not. It appears that decimals simply do not align well with ratios defined over discrete visual quantities; thus allowing extra time does not eliminate their disadvantage in accuracy. Fractions with a numerator and denominator that are proportional but not equal to the depicted quantities (NOTO fractions) add extra processing time relative to fractions that have a one-to-one (OTO) mapping. But in the absence of speed pressure, even NOTO fractions yield greater accuracy in analogical reasoning with discrete displays than do decimals.

General Discussion

Summary

The present study, to the best of our knowledge, provides the first evidence that the internal structure of an individual number can provide a model of relations in the external environment, thereby altering performance in tasks that require reasoning about these relations. Specifically, we tested the hypotheses that fractions are conceptually linked to countable discrete entities, and naturally express a two-dimensional relationship between the cardinal values of sets; whereas decimals are conceptually linked to continuous masses, and more naturally express the relative magnitude of a proportional relation. Previous work has shown that although decimals and fractions can express equivalent magnitudes (subject to rounding error on decimals), the one-dimensional nature of decimals is, in fact, advantageous in representing mag-

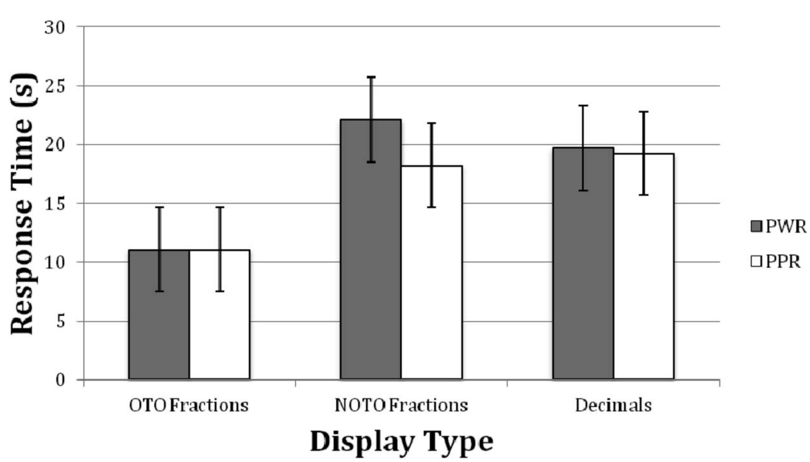

Figure 14. Mean response time for analogical inference using fractions and decimals across different types of displays (Experiment 4B). Error bars indicate standard error of the mean.

nitudes (DeWolf et al., 2014). Here, we found that the bipartite notation of fractions is advantageous in reasoning about ratio relations (either part-to-whole or part-to-part).

Our experiments used visual displays showing either two countable subsets or two parts of a continuous area. Experiment 1 demonstrated that college-educated adults prefer to use fractions to represent ratio relations between countable sets, and prefer to use decimals to represent ratio relations of continuous masses, in a task that does not require any mathematical procedures. Hence, these findings indicate that the selective affinity of fractions with discrete entities and decimals with continuous entities has a conceptual basis.

Experiments 2–4 examined relational reasoning with fractions and decimals when procedural computations were required. The bipartite structure of fractions (*a/b*) invites counting the size of two separate sets, whereas the decimal notation invites an estimate of the one-dimensional magnitude of a ratio—a procedure that does not depend on discrete elements, but that is less accurate than counting discrete elements. The results of Experiments 2–4 demonstrated an overall advantage for fractions over decimals in relational tasks based on ratios—both relation identification (Experiment 2) and higher-order analogical reasoning (Experiments 3 and 4). However, this advantage was moderated by the nature of the depicted quantities. Fractions allowed more accurate relational reasoning when the depicted quantities were discrete elements, or continuous quantities that had been discretized by introducing units suitable for measurement. This fraction advantage reflects the fact that fractions align well with discrete quantities, which, in turn, support exact calculation procedures, such as counting. Fractions may still encourage counting (by mental "slicing" into units) even for continuous displays (as suggested by the regression analyses performed in Experiment 2), though accuracy is reduced. Performance with decimals was relatively equal, and less accurate, for all quantity types, suggesting that decimals are preferentially evaluated using estimation rather than counting. Although decimals naturally align with continuous quantities and fractions do not (as shown in Experiment 1), decimals support less accurate computational procedures in these relational tasks, and hence are no more effective than fractions when such tasks require performing computations on continuous quantities.

Alternative Interpretations

Several "deflationary" accounts of the present findings deserve consideration. It might be argued that the observed differences in reasoning with fractions versus decimals are simply another example of the general phenomenon that alternative notation systems for number provide dramatically different algorithmic affordances (e.g., computing 73×27 using Arabic numerals is considerably easier than LXXIII × XXVII using Roman numerals; see Zhang & Norman, 1995, for a general analysis of number systems used in different cultures). However, the differences in the affordances offered by fractions and decimals do not involve comparisons between notations drawn from different cultures and historical periods. Rather, fractions and decimals are both familiar number types defined within the Arabic system, in common use throughout the world today. The present findings show that even within the basic number system in near-universal use in the modern world, different number formats vary in their affordances for both calculation and reasoning.

It also might be argued that our findings simply show that discrete quantities elicit counting, a procedure associated with fractions, whereas continuous quantities elicit magnitude estimation, which is associated with decimals. This is indeed a reasonable summary of much of our findings, but we believe the empirical phenomena are more meaningful when placed in a theoretical framework based on the properties of symbolic notations as models. First of all, it is by no means obvious a priori that fractions are necessarily associated with discrete representations. Fractions might be interpreted as holistic numbers (Schneider & Siegler, 2010), providing a possible basis for them to align with continuous quantities. More generally, the primary focus of psychological research on numerical cognition in recent years has been on magnitude representation, the common property associated with all number concepts. However, the present findings do not show evidence that adults process fractions holistically. Rather, componential processing of the magnitudes associated with the numerator and denominator appeared to be sufficient to identify the relations, and thus seemed to provide the preferred strategy. Furthermore, adults did not seem to adopt a holistic strategy even for continuous displays, as demonstrated by the poorer performance for fractions with such displays.

The framework we have presented here serves to call attention to another basic property of numerical systems—the representation of quantitative relations and procedures for reasoning with them. An analysis of the internal structure of rational numbers makes it clear why fractions are especially suited for reasoning about relations between the cardinality of sets, whereas decimals are better-suited for magnitude comparisons. We suspect that the relational structure of fractions is closely linked to the acquisition of more complex relational concepts involved in algebra, and hence has important implications for instruction (as elaborated below).

The high-level summary of our findings also misses important nuances in our data. In particular, Experiment 1 showed that adults preferentially associate discrete displays with fractions, and continuous displays with decimals, even when the task does not require any kind of computation. Thus, the conceptual linkage between fractions and decimals with discrete and continuous quantities, respectively, holds even when neither counting nor magnitude estimation is required. (Below we discuss implications of our framework for reasoning tasks involving natural frequencies, which also do not require counting.) Indeed, the pattern of results changed in an important way in the later experiments, for which computation was critical to perform the task. In these reasoning tasks, people showed an advantage for fractions with discrete displays, but relatively equal (and poorer) performance for both symbol types with continuous displays. The lack of a decimal advantage for continuous displays when computation is required can best be understood by taking account not only of the preferred associations between symbol types and procedures, but also of the relative precision of those procedures. Finally, Experiments 4A and 4B showed that the advantage of fractions over decimals for discrete displays is obtained even when the numerator and denominator of the fraction do not match the cardinality of the relevant sets, as long as the proportional relation is maintained. Thus, the strategy underlying the fraction advantage is more complex than a simple "count and match."

Natural Frequencies as Fraction-Like Representations

The theoretical framework we have presented has the advantage of connecting work on numerical cognition with other research areas related to quantitative reasoning. In particular, the present evidence that fractions can function as relational models provides insight into other findings indicating that people's accuracy in relational reasoning depends on the format of rational numbers. For example, studies of tasks that require Bayesian inference (in particular, integration of base rates with likelihoods) have consistently found an advantage for natural frequency formats over probabilities, percentages, and other formats that have been "simplified" by removing or standardizing the size of the specified population (Gigerenzer & Hoffrage, 1995; Hoffrage et al., 2002; Tversky & Kahneman, 1983). For example, observing that 40 of 1,000 people have a certain disease can be summarized as a natural frequency, 40/1000. As a type of fraction, the numerator and denominator in this bipartite frequency format align with the sizes of the subset and the population, respectively. In the terminology of the present study, a natural frequency is a fraction with a one-to-one relationship to a part-to-whole ratio (a symbolic notation tested in all our reported experiments). In contrast, reduced expressions equivalent in magnitude, such as a decimal representing the proportion (0.04), or a percentage (4%), are one-dimensional, expressing the magnitude of the natural frequency but not its internal structure (Halford et al., 1998, 2010). Moreover, as Hoffrage et al. (2002) correctly emphasized, relative frequencies that use a standardized denominator, such as 100, also lose information about the size of the population. For example, although the natural frequency 40/1000 is equivalent in magnitude to the standardized relative frequency 4/100, the denominator in the latter is a constant rather than a variable, which means that the dimensionality of the structure has been reduced from two to one. Though the magnitude has been preserved, the relational structure has been obscured.

Based on a literature review, Hoffrage et al. (2002) report that Bayesian inference tends to be more accurate with natural frequencies than with any of the various one-dimensional notations (including standardized relative frequencies) that express equivalent magnitudes. Importantly, the frequencies in these tasks were directly stated to participants as summary statistics; no counting or estimation was required. Gigerenzer and Hoffrage (1995) interpret the natural frequency advantage in terms of the evolutionary origins of frequency information in animal activities such as foraging. However, since all explicit number notations are human cultural artifacts, the connection to biological evolution would appear to be tenuous (see Barbey & Sloman, 2007, for discussion of alternative hypotheses). From our perspective, the proximal basis for the advantage of natural frequencies in supporting certain types of inferences is simply that their format affords closer alignment between the mathematical model and the structure of relevant relations in the world (as was also the case for the relational reasoning tasks investigated in the present study). This interpretation is consistent with the hypothesis that natural frequencies are particularly good cues to the structure of problems involving nested sets (e.g., "10 out of 1,000 smokers in the sample developed lung cancer"; see Tversky & Kahneman, 1983; Barbey & Sloman, 2007). For different reasoning tasks (notably, magnitude comparison) where the set structure is not critical (and, indeed, may be a distraction), we would expect other formats (e.g., proportions) to be more effective than natural frequencies (for an example, see Over, 2007). Thus, although the present study involved reasoning with simplified visual displays devoid of meaning, our findings have implications for real-world reasoning tasks involving relations between the cardinalities of sets.

Approximate Estimation Versus Exact Calculation

The present findings are consistent with the hypothesis that decimals invite an approximate strategy for estimating magnitudes of ratios (Jacob et al., 2012). For continuous quantities, this estimation strategy is about as effective as the best strategy available for fractions. However, for discrete displays that enable exact calculations, fractions support a more accurate strategy. When the numbers forming the fraction match the quantities in the display, decisions based on fractions were more accurate than those based on decimals, with no cost in response time. Decisions for fractions were slower when their constituent numbers were not in one-to-one correspondence with the displayed quantities (NOTO fractions; Experiment 4A). However, NOTO fractions (which preserve proportionality though not equality with the displayed quantities) still yielded greater accuracy than decimals, even when time pressure was eliminated and response times equated (Experiment 4B). The fraction advantage thus extends beyond the benefit of a direct match between the numbers in the fraction and the numbers of entities in the displays. Rather, fractions have a more basic advantage attributable to the ease of aligning their bipartite structure with the natural situation model. As long as the depicted quantities are countable, fractions enable use of a more precise computation strategy, yielding greater accuracy than the estimation strategy encouraged by decimals.

It should be emphasized that the accuracy advantage afforded by exact computation strategies depends on acquiring competence in the necessary computation. For our college-student participants, exact computation of relatively small discrete quantities was presumably a well-learned skill. But because exact computation is acquired later than the more elementary approximation strategies, young children may make less accurate judgments when quantities are discretized rather than continuous (Boyer et al., 2008). The paradigms introduced in the present study may provide useful diagnostic tools for tracking developmental changes in reasoning with rational numbers.

The present findings lend support to the hypothesis that fractions are not automatically processed as a holistic magnitude. In the reasoning tasks we assessed, for which holistic processing is not beneficial, participants were able to perform well using componential processing of fractions. Previous research on magnitude representations of fractions has shown that fractions can indeed be processed holistically (Schneider & Siegler, 2010), but this process in not automatic (DeWolf et al., 2014) and is not necessarily preferred over componential processing (Bonato et al., 2007). Our results support the hypothesis that the processing of fractions is task-dependent. Adults often seem to be able to adopt whatever strategy for processing fractions is most suitable for the task at hand.

In many instances of mathematical problem solving, it is not important for students to actually calculate or find a magnitude in order to solve a problem. Rather, simply understanding the relation is sufficient, and sometimes can be more useful and adaptive. For example, when deciding which is larger, $x/5$ versus $x/6$, knowing the actual magnitude of x is impossible. To find the correct answer, it is sufficient to note that the same number, x, is divided either by a larger number, 6, or a smaller number, 5. If one knows that division by larger numbers results in smaller numbers (assuming x is positive), it follows that $x/5$ is larger than $x/6$. Thus, the answer can be inferred by using relational knowledge about relative sizes, division, and equivalence, without any direct assessment of magnitudes. Similarly, if one knows that $m \times (3/5) = n$, one can infer that $n \times (5/3) = m$ without any information about the magnitudes of m and n, simply by understanding multiplication and reciprocal relations. For higher levels of mathematics, such as algebra and calculus, problems are often solved not by identifying specific magnitudes, but by understanding mathematical relationships.

Implications for Teaching Rational Numbers

The present findings, in conjunction with previous research on processing of fractions and decimals, may have implications for how these types of symbolic notations could be most effectively taught. The present analysis and findings shed light on why fractions are especially difficult for students to learn (Siegler, Fazio, Bailey, & Zhou, 2013; Ni & Zhou, 2005; Stigler, Givvin, & Thompson, 2010; Vamvakoussi & Vosniadou, 2004), and also are especially important predictors of success in acquiring more advanced mathematical knowledge in high school (Siegler et al., 2011, 2012). In the United States, fractions are generally the first number type introduced in school after the familiar natural numbers. Whereas natural numbers fundamentally express a unidimensional magnitude, fractions are two-dimensional. Not only is their bipartite format unfamiliar, but their conceptual structure is inherently more complex than that of natural numbers, placing greater demands on working memory (English & Halford, 1995; Halford et al., 1998, 2010). Although fractions indeed represent magnitudes, magnitudes do not fully capture their meaning. Elementary-school students (who are likely to be unevenly developed with respect to working memory capacity) may have difficulty grasping the two-dimensional structure of fractions, particularly if instruction focuses on their magnitudes rather than on their relational meaning. But in many reasoning tasks, including those investigated in the present study, the two-dimensional structure of fractions can, in fact, be exploited to advantage. Although fractions are relatively inefficient as representations of magnitudes, they can be very effective as representations of relations.

Fractions thus have a dual status that poses particular challenges for students: a fraction is at once a relationship between two quantities, expressed as a/b, and also the magnitude corresponding to the division of a by b. Gray and Tall (1994) have argued that children's understanding of arithmetic is dependent on their "proceptual" understanding: grasping that a mathematical expression containing an arithmetic operation embodies the process of obtaining a certain result (similar to the "process-product dilemma" in algebra discussed by Sfard & Linchevski, 1994). Gray and Tall (1994) found that students who become proficient in arithmetic at an earlier age show greater ability to move back and forth flexibly between an arithmetic expression and the result of that expression. The difficulty in understanding a fraction as a relational expression may explain why young children appear to understand quantitative relations such as part-whole or proportions with visual displays (Goswami, 1989; Mix, Levine & Huttenlocher, 1999; Boyer, Levine, & Huttenlocher, 2008; Sophian, 2000), but not when they have to answer comparable questions with fraction notation (Ball & Wilson, 1996; Mack, 1995). For example, Ni and Zhou (2005) found that most children could answer the question, "How much is one third plus one third?" verbally as "two thirds." But when asked the question using symbolic fraction notation, "1/3 + 1/3 = ?", most children answered "2/6" (often claiming that both 2/6 and 2/3 are correct answers). Children thus seem to have some intuitive understanding of how rational quantities relate to one another, but have difficulty understanding the novel symbolic expressions.

It is therefore important to convey the dual nature of fractions to students, with a focus on their relational structure. The multidimensional structure of numbers eventually proves to be critical in understanding even the simplest algebraic expressions, such as $(2/3)x$. Thus, those students who eventually succeed in grasping the very concept of a multidimensional number, first instantiated by common fractions, will have mastered a fundamental prerequisite for advanced mathematics.

The conceptualization of fractions as relations such as part/whole, subset/set, ratio and proportions may have implications for how children are able to solve problems using fractions in these contexts. Typically, school instruction in the United States only emphasizes the part-to-whole relationship (Sophian, 2007; Mack, 1993), which most clearly relates to the understanding of fractions as magnitudes. Children are first introduced to fractions using pictorial representations intended to help students understand the meaning of a value smaller than one. As we observed earlier, magnitudes are far easier to process using one-dimensional decimals than bipartite fractions, even for adults (DeWolf et al., 2014). Thus, it might well be easier for children to learn about magnitudes less than one by being introduced to decimals prior to fractions. Fractions might be taught later than decimals, with an emphasis on their status as a *relationship* between two natural numbers and that multiple possible relationships can be of interest. For example, teaching students about the part-to-part relation in addition to the part-to-whole relation might help to expand children's understanding of fractions. This more general understanding might, in turn, aid students in eventually learning more abstract mathematics, such as algebra.

In fact, Moss and Case (1999) implemented a curriculum with 4th graders in Canada that reorganized the usual order of instruction in rational numbers. Children were first taught percentages (in the context of volumes, and on number lines), then decimals, and lastly fractions. Fractions were explained simply as another way to represent a decimal. By contrast, typical curricula describe teaching decimals as another way to represent a fraction. Moss and Case found that children taught the symbolic notations in this novel sequence suffered less interference from whole-number strategies when using other rational numbers, and achieved a deeper understanding of them. This approach to teaching rational numbers is supported by our framework, as this instructional method encourages students to first master the concept of rational-number magnitude with the symbolic notation that more effectively represents such magnitude: decimals. Though Moss and Case did not emphasize the role of fractions in the types of relational contexts we have discussed here, this type of curriculum could be used to allow students to master decimal magnitudes and then later learn about fractions in the context of modeling types of relations. Thus, a major issue that arises for students using current curricula in the U.S.—understanding fraction magnitudes—might be de-emphasized in this context.

More generally, understanding how types of rational numbers align to specific types of quantities, and how the internal structure of mathematical expressions can affect ease of alignment to the perceptual or semantic relations being modeled, has important implications for how to best conceptualize and teach fractions and decimals. It is important to foster understanding of mathematics in a way that goes beyond teaching algorithmic procedures. A mathematical expression can represent not just a procedure, and not just a magnitude, but also a relational structure that maps onto the structure of the world.

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(*Appendix follows*)

Appendix

Experimental Stimuli

Table A1
Ratios Used in Experiment 1

Continuous	Discretized	Discrete
1/10	2/11	2/9
1/6	2/10	2/7
1/4	2/9	2/10
4/13	2/7	3/10
3/6	2/6	3/9
3/9	4/13	3/11
4/11	5/15	3/8
5/12	4/10	4/10
3/7	5/12	4/14
4/9	3/7	3/7
5/11	5/9	3/4
5/9	4/7	4/9
5/14	7/12	5/10
4/7	5/8	6/11
6/10	5/13	4/7
9/13	4/6	6/10
8/10	7/9	2/3
5/6	8/10	7/10
7/8	7/8	8/10
8/15	4/5	4/6

Note. In Experiment 1, participants were not told the actual ratio given, nor were they asked to identify this ratio. The numbers in this table indicate the number of pieces in the red and green subsets for the various displays.

Table A2
Ratios Used in Experiment 2

	PPR		PWR	
	Fraction	Decimal	Fraction	Decimal
Discrete	5/7	.71	7/10	.70
	4/6	.67	10/15	.67
	6/10	.60	5/6	.83
	6/8	.75	7/8	.88
	4/8	.50	4/7	.57
	4/5	.80	6/11	.55
	2/7	.29	2/9	.22
	3/9	.33	2/5	.40
	4/9	.44	5/15	.33
	3/8	.38	3/10	.30
	4/10	.40	4/14	.29
	3/7	.43	3/7	.43
Discretized	6/8	.75	9/10	.90
	7/8	.88	5/8	.63
	5/8	.63	7/12	.58
	5/9	.56	4/6	.67
	4/5	.80	7/9	.78
	4/7	.57	4/5	.80
	3/7	.43	4/13	.31
	2/9	.22	2/9	.22
	2/10	.20	4/10	.40
	2/7	.29	2/7	.29
	1/4	.25	5/12	.42
	3/9	.33	1/9	.11
Continuous	4/5	.80	9/16	.56
	4/7	.57	7/8	.88
	7/8	.88	3/5	.60
	6/8	.75	9/13	.69
	5/6	.83	11/14	.79
	5/9	.56	9/15	.60
	4/13	.31	1/10	.10
	5/15	.33	6/13	.46
	4/9	.44	3/9	.33
	2/8	.25	6/14	.43
	4/11	.36	4/11	.36
	5/12	.42	2/12	.17

(*Appendix continues*)

Table A3
Ratios Used for Target Displays in Experiment 3

	PPR				PWR			
	Fraction		Decimal		Fraction		Decimal	
	Target	Foil	Target	Foil	Target	Foil	Target	Foil
Discrete	3/5	3/8	.60	.38	3/7	3/4	.43	.75
	6/7	6/13	.86	.46	2/5	2/3	.40	.67
	5/9	5/14	.56	.36	3/8	3/5	.38	.60
	3/10	3/13	.30	.23	1/6	1/5	.17	.20
	3/7	3/10	.43	.30	4/13	4/9	.31	.44
	2/9	2/11	.22	.18	2/12	2/10	.17	.20
	4/7	4/11	.57	.36	4/9	4/5	.44	.80
	2/8	2/10	.25	.20	5/13	5/8	.38	.63
	6/8	6/14	.75	.43	6/14	6/8	.43	.75
	4/10	4/14	.40	.29	3/11	3/8	.27	.38
	2/6	2/8	.33	.25	2/8	2/6	.25	.33
	4/9	4/13	.44	.31	2/11	2/9	.18	.22
Discretized	5/7	5/12	.71	.42	5/12	5/7	.58	.71
	3/4	3/7	.75	.43	6/13	6/7	.46	.86
	3/5	3/8	.60	.38	5/11	5/6	.45	.83
	2/5	2/7	.40	.29	2/11	2/9	.18	.22
	3/10	3/13	.30	.23	3/10	3/7	.30	.43
	5/11	5/16	.45	.31	2/12	2/10	.17	.20
	7/8	7/15	.88	.47	3/7	3/4	.43	.75
	4/6	4/10	.67	.40	5/13	5/8	.38	.63
	7/8	7/15	.88	.47	4/9	4/5	.44	.80
	3/11	3/14	.27	.21	4/13	4/9	.31	.44
	2/12	2/14	.17	.14	2/9	2/7	.22	.29
	1/5	1/6	.20	.17	2/14	2/12	.14	.17
Continuous	4/6	4/10	.67	.60	7/15	7/8	.47	.88
	6/7	6/13	.86	.46	3/11	3/8	.27	.38
	6/11	6/17	.55	.35	7/16	7/9	.44	.78
	2/9	2/11	.22	.18	2/14	2/12	.14	.17
	3/8	3/11	.38	.27	4/17	4/13	.24	.31
	3/16	3/19	.19	.16	2/7	2/5	.29	.40
	7/12	7/19	.58	.37	6/13	6/7	.46	.86
	5/6	5/11	.83	.45	7/17	7/10	.41	.70
	7/9	7/16	.78	.44	8/18	8/10	.44	.80
	4/13	4/17	.31	.24	4/14	4/10	.29	.40
	2/6	2/8	.33	.25	3/14	3/11	.21	.27
	4/10	4/14	.40	.29	2/11	2/9	.18	.22

Note. Each of these target and foil combinations were randomly matched with a source picture of the same entity type and ratio type from Experiment 2 (as shown in Table A2).

(*Appendix continues*)

Table A4
Ratios Used for Source Displays in Experiments 4A and 4B

PPR			PWR		
OTO fraction	NOTO fraction	Decimal	OTO fraction	NOTO fraction	Decimal
5/7	10/14	.71	7/10	14/20	.70
4/6	2/3	.67	10/15	2/3	.67
6/10	3/5	.60	5/6	10/12	.83
6/8	3/4	.75	7/8	14/16	.88
4/8	2/4	.50	4/7	12/21	.57
4/5	8/10	.60	6/11	12/22	.55
2/7	4/14	.29	2/9	4/18	.22
3/9	1/3	.33	2/5	6/15	.40
4/9	8/18	.44	5/15	1/3	.33
3/8	6/16	.38	3/10	6/20	.30
4/10	2/5	.40	4/14	2/7	.29
3/7	6/14	.43	3/7	6/14	.43
6/11	12/22	.55	9/10	18/20	.90
7/8	14/16	.88	5/8	10/16	.63
5/8	10/16	.63	7/12	14/24	.58
5/9	10/18	.56	4/6	8/12	.67
12/14	6/7	.86	7/9	14/18	.78
4/7	8/14	.57	4/5	12/15	.80
4/11	8/22	.36	4/13	8/26	.31
2/9	4/18	.22	2/9	6/27	.22
2/10	1/5	.20	4/10	2/5	.40
3/10	6/20	.30	2/7	6/21	.29
1/4	3/12	.25	5/12	10/24	.42
1/8	3/24	.13	1/9	3/27	.11

Note. In Experiments 4A and 4B all displays were discretized.

(*Appendix continues*)

Table A5
Ratios Used for Target Displays in Experiments 4A and 4B

PPR						PWR					
OTO fraction		NOTO fraction		Decimal		OTO fraction		NOTO fraction		Decimal	
Target	Foil	Target	Foil	Target	Foil	Target	Foil	Target	Foil	Target	Foil
3/5	3/8	6/10	6/16	.60	.38	3/7	3/4	6/14	6/8	.43	.75
6/7	6/13	12/14	12/26	.86	.46	2/5	2/3	4/10	4/6	.40	.67
5/9	5/14	10/18	10/28	.56	.36	3/8	3/5	6/16	6/10	.38	.60
3/10	3/13	6/20	6/26	.30	.23	1/6	1/5	3/18	3/15	.17	.20
3/7	3/10	6/14	6/20	.43	.30	4/13	4/9	8/26	8/18	.31	.44
2/9	2/11	4/18	4/22	.22	.18	2/12	2/10	1/6	1/5	.17	.20
4/7	4/11	8/14	8/22	.57	.36	4/9	4/5	8/18	8/10	.44	.80
2/8	2/10	1/4	1/5	.25	.20	5/14	5/8	10/26	10/16	.38	.63
6/8	6/14	3/4	3/7	.75	.43	6/14	6/8	3/7	3/4	.43	.75
4/10	4/14	2/5	2/7	.40	.29	3/11	3/8	6/22	6/16	.27	.38
2/6	2/8	1/3	1/4	.33	.25	2/8	2/6	1/4	1/3	.25	.33
4/9	4/13	8/18	8/26	.44	.31	2/11	2/9	4/22	4/18	.18	.22
5/7	5/12	10/14	10/16	.71	.42	5/12	5/7	10/24	10/14	.42	.71
3/4	3/7	6/8	6/14	.75	.43	6/13	6/7	12/26	12/14	.46	.86
4/5	4/9	8/10	8/18	.80	.44	5/11	5/6	10/22	10/12	.45	.83
2/5	2/7	4/10	4/14	.40	.29	4/14	4/10	2/7	2/5	.29	.40
2/7	2/9	4/14	4/18	.29	.22	3/10	3/7	6/20	6/14	.30	.43
5/11	5/16	10/22	10/32	.45	.31	2/12	2/10	1/6	1/5	.17	.20
7/8	7/15	14/16	14/30	.88	.47	5/15	5/9	10/28	10/18	.36	.56
4/6	4/10	2/3	2/5	.67	.40	4/12	4/8	2/6	2/4	.33	.50
8/9	8/17	16/18	16/34	.89	.47	4/9	4/5	8/18	8/10	.44	.80
3/11	3/14	6/22	6/28	.27	.21	7/15	7/8	14/30	14/16	.47	.88
2/12	2/14	1/6	1/7	.17	.14	2/9	2/7	4/18	4/14	.22	.29
1/5	1/6	3/15	3/18	.20	.17	2/14	2/12	1/7	1/6	.14	.17

Note. In Experiments 4A and 4B all displays were discretized.

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